Generalizations of BRST symmetry

We have an action $I[\phi]$ and measure $[d\phi] = \prod_r d\phi^r$ that are invariant under the generalized symmetry

$$\phi^r \to \phi^r + \varepsilon^A \delta_A \phi^r \qquad (1)$$

We are using "De Witt" notation: r and A include spacetime coordinates as well as discrete labels

Example:

Consider the gauge transformation

$$\delta A^\beta{}_\mu = \partial_\mu \varepsilon^\beta + C^\beta{}_{\gamma\alpha} \varepsilon^\alpha A^\gamma{}_\mu \qquad (2)$$

$\rightarrow$ we have $r = (\mu, \alpha, x)$ so $\phi^{\mu\alpha x} \equiv A^\alpha{}_\mu(x)$

and $A = (\alpha, x)$ so $\varepsilon^{\alpha x} \equiv \varepsilon^\alpha(x)$ in the notation (1)

$\rightarrow$ (2) becomes

$$\delta_{\beta y} \phi^{\mu\alpha x} = \delta^\beta_\alpha \frac{\partial \delta^4(x-y)}{\partial x^\mu} + C^\beta{}_{\gamma\alpha} \phi^{\mu\gamma x} \delta^4(x-y)$$

Check:

$$\varepsilon^A \delta_A \phi^r = \int dy \; \varepsilon^\alpha(y) \, \delta^\beta_\alpha \underbrace{\frac{\partial}{\partial x^\mu} \delta^4(x-y)}_{= -\frac{\partial}{\partial y^\mu} \delta^4(x-y)} + \varepsilon^\alpha(y) C^\beta{}_{\gamma\alpha} \phi^{\mu\gamma x} \delta^4(x-y)$$

$$= \partial_\mu \varepsilon^\beta + \varepsilon^\alpha(x) A^\gamma{}_\mu(x) C^\beta{}_{\gamma\alpha} \quad \checkmark$$

Generalized BRST symmetry can be derived from general Faddeev-Popov-De Witt theorem:

$$\frac{C}{\Omega}\int [d\phi]\, e^{iI[\phi]} V[\phi] = \int [d\phi]\, e^{iI[\phi]} \mathcal{B}[f[\phi]]\, \mathrm{Det}\left(\delta_A f_B[\phi]\right) V[\phi] \quad (3)$$

where $V[\phi]$ is arbitrary functional of ϕ^r invariant under the symmetry (1),

$f_A[\phi]$: gauge fixing functionals

$\mathcal{B}[f]$: arbitrary functional of the f_A

Ω : volume of the gauge group

$C \equiv \int [df]\, \mathcal{B}[f]$

$\rightarrow$ equation (3) tells us that integral on right-hand side is independent from choice of gauge-fixing functionals f_A

Now express $\mathcal{B}[f]$ as a Fourier transform

$$\mathcal{B}[f] = \int [dh] \exp(i h^A f_A)\, \mathcal{B}[h],$$

where $[dh] \equiv \prod_A dh^A$. Furthermore,

$$\mathrm{Det}(\delta_A f_B[\phi]) \sim \int [d\omega^*][d\omega] \exp(i\omega^{*B}\omega^A \delta_A f_B),$$

where $[d\omega^*] \equiv \prod_A d\omega^{*A}$ and $[d\omega] \equiv \prod_A d\omega^A$

$$\rightarrow \int [d\phi] \exp(iI[\phi])\, \mathcal{B}[f[\phi]]\, \mathrm{Det}(\delta_A f_B[\phi])\, V[\phi]$$

$$\sim \int [d\phi][dh][d\omega^*][d\omega] \exp\left(iI_{NEW}[\phi, h, \omega, \omega^*]\right) \mathcal{B}[h]\, V[\phi]$$

where $I_{NEW}[\phi, h, \omega, \omega^*] = I[\Phi] + h^A f_A[\Phi] + \omega^{*B} \omega^A \delta_A f_B[\Phi]$

(3)

<u>Remark:</u>

ghost fields are compensation for integrating over too many degrees of freedom (gauge trfs.)

ghosts are fermions $\rightarrow$ loops carry minus sign

$\rightarrow$ cancels contribution of gauge equivalent Φ's

I_{NEW} has an extra symmetry under BRST trfs.

$$\chi \mapsto \chi + \theta s \chi \qquad (4)$$

where χ is any of the $\phi^r, \omega^A, \omega^{A*}$, or h^A; θ is an infinitesimal anti-commuting c-number; and s is the "Slavnov operator":

$$s = \omega^A \delta_A \phi^r \frac{\delta_L}{\delta \phi^r} - \frac{1}{2} \omega^B \omega^C f^A{}_{BC} \frac{\delta_L}{\delta \omega^A} - h^A \frac{\delta_L}{\delta \omega^{*A}}$$

superscript L denotes left differentiation:

$$\delta F = \delta \chi G \longrightarrow \delta_L F / \delta \chi = G$$

and $f^A{}_{BC}$ is the structure constant appearing in $[\delta_B, \delta_C] = f^A{}_{BC} \delta_A$ (*)

$f^A{}_{BC}$ are field independent in non-abelian gauge theories and string theories, but more general situation $f^A{}_{BC}[\Phi]$ possible.

We compute

$$s^2 = \frac{1}{2}\omega^A\omega^B \Big[\underbrace{\delta_A\phi^s \frac{\delta_L(\delta_B\phi^r)}{\delta\phi^s} - \delta_B\phi^s \frac{\delta_L(\delta_A\phi^r)}{\delta\phi^s} - f^C{}_{AB}\delta_C\phi^r}_{=0 \text{ by } (*)}\Big]\frac{\delta_L}{\delta\phi^r}$$

$$- \frac{1}{2}\omega^B\omega^C\omega^D\Big[f^E{}_{BC}f^A{}_{DE} + \delta_D\phi^r \frac{\delta_L f^A{}_{BC}}{\delta\phi^r}\Big]\frac{\delta_L}{\delta\omega^A}$$

→ consistency condition for vanishing of s^2:

$$f^E{}_{[BC}f^A{}_{D]E} + \delta_{[D}\phi^r\left(\delta_L f^A{}_{BC]}/\delta\phi^r\right) = 0$$

→ equivalent to Jacobi identity for field independent $f^A{}_{BC}$.

Let's check that (4) is a symmetry of (3):

rewrite (3) as

$$I_{NEW}[\phi, h, \omega, \omega^*] = I[\phi] - s(\omega^{*A} f_A)$$

$I[\phi]$ is BRST invariant as on the fields ϕ^r a BRST transformation is just a gauge trf. with $\varepsilon^A = \theta\omega^A$!

Also $s(s(\omega^{*A}f_A)) = ss(\omega^* f_A) = 0$

Most general BRST-invariant functional:

$$I_{NEW}[\phi, \omega, \omega^*, h] = I_0[\phi] + s\Psi[\phi, \omega, \omega^*, h]$$

Proof:

a) Note that BRST-trf. does not change total number of h^A and ω^{*A} fields

$\rightarrow$ expand $I = \sum_N I_N$ ← contains definite # of h^A and ω^{*A} fields

and we have $sI_N = 0$ $\quad (**)$

b) introduce "Hodge operator":

$$t \equiv \omega^{*A} \frac{\delta}{\delta h^A}$$

Then we have

$$[s,t]_+ = st + ts = s\left(\omega^{*A} \frac{\delta_L}{\delta h^A}\right) - t\left(h^A \frac{\delta_L}{\delta \omega^{*A}}\right)$$

$$= -h^A \frac{\delta_L}{\delta h^A} - \omega^{*A} \frac{\delta_L}{\delta h^A}$$

$$\overset{(**)}{\Rightarrow} st\, I_N = -N I_N$$

Thus each I_N except for I_0 is BRST-exact.

$$\rightarrow I = I_0 + s\Psi$$

where $\Psi = -\sum_{N=1}^{\infty} \frac{t I_N}{N}$

I_0 is independent of ω^{*A}, h^A, and ω^A (ghost # = 0). $\square$

Invariance of physical matrix elements:

Define "charge" Q by

$$\delta_\theta \Phi = i\,[\theta Q, \Phi] = i\theta [Q, \Phi]_\mp$$

with $[X,Y]_\mp \equiv XY \mp YX$ according to as Φ is bosonic or fermionic.

nilpotence $\longrightarrow$ $Q^2 = 0$

Matrix elements will be independent of choice of Ψ iff $Q|\alpha\rangle = \langle\beta|Q = 0$

$\rightarrow$ {physical states}

= {elements of Q-cohomology}

general recepie: find <u>some</u> Ψ (like axial gauge), in which the ghosts do not interact with other fields $\rightarrow$ from there choose <u>any</u> Ψ convenient for computation. Then ghost-free initial and final states